Problem Set 9: NP-completeness

1. (Easy) Show that Vertex Cover $\leq_p$ Independent Set

2. (Easy) Let VC-Dec be the decision version of the vertex cover problem and VC-Opt be its optimisation version. Show that VC-Dec $\leq_p$ VC-Opt and VC-Opt $\leq_p$ VC-Dec

3. (Easy) TSP problem: Given a matrix of distances between each pair of cities and a budget b, determine whether there exists a tour that visits every city exactly once, returns to the starting city, and has total length at most b.

   Show that if TSP can be solved in polynomial time, then the optimization version (TSP-OPT), which asks for the shortest such tour, can also be solved in polynomial time.

4. (Easy) Given an arbitrary graph $G = (V, E)$, suppose you have an algorithm that returns, in polynomial time, the size of the largest complete subgraph of G (i.e., the size of a maximum clique).

   Design and analyze a polynomial-time algorithm that, given an arbitrary graph G, outputs a *complete subgraph* of G of maximum size, using this algorithm as a subroutine. Your algorithm should compute the actual set of vertices that form the maximum clique in G.

5. (Easy) k-HP: Given an undirected graph $G = (V, E)$ and a number k, the problem asks: "Does G contain a simple path that visits at least $|V| - k$ vertices?"

   Hamiltonian Path Problem (HP): Given a graph $G = (V, E)$, does there exist a path that visits each vertex exactly once?

   Show that HP $\leq_p$ k-HP

6. (Easy)

   Partition Problem: Partition

   Given a set of integers $S = \{s_1, s_2, \ldots, s_n\}$, does there exist a partition of S into two subsets S_1 and S_2 such that

   $$\sum_{s \in S_1} s = \sum_{s \in S_2} s?$$

   Subset Sum Problem: Subset

   Given a set of integers $S = \{s_1, s_2, \ldots, s_n\}$ and an integer t, does there exist a subset $S' \subseteq S$ such that

   $$\sum_{s \in S'} s = t?$$

   Show that Partition $\leq_p$ Subset and Subset $\leq_p$ Partition

7. (Medium) In the HITTING SET problem, we are given a family of sets $\{S_1, S_2, \ldots, S_n\}$ and a budget b, and we wish to find a set H of size $\leq b$ which intersects every S_i, if such an H exists. In other words, we want $H \cap Si = \emptyset$ for all i. Show that HITTING SET is NP-complete. [Hint: You can assume that Set Cover is NP-Complete. Try to reduce Set Cover to Hitting Set]

8. (Medium) Integer-Programming

   Given an $m \times n$ matrix A and an m-dimensional vector b, is there an n-dimensional vector x with coefficients $x_i \in \{0, 1\}$ such that:

   $$Ax \geq b$$

   Show that 3-SAT $\leq_p$ Integer-Programming

9. (MEDIUM) A boolean formula is in **disjunctive normal form** (or DNF) if it consists of a disjunction ($\vee$) of several terms, each of which is the conjunction ($\wedge$) of one or more literals. For example, the formula

$$(\bar{x} \wedge y \wedge z) \vee (\bar{y} \wedge z) \vee (\bar{x} \wedge y \wedge z)$$

is in disjunctive normal form. **DNF-SAT** asks, given a boolean formula in disjunctive normal form, whether that formula is satisfiable.

(a) Describe a polynomial-time algorithm to solve DNF-SAT.

(b) What is the error in the following argument that $P = NP$?

Suppose we are given a boolean formula in conjunctive normal form with at most three literals per clause, and we want to know if it is satisfiable. We can use the distributive law to construct an equivalent formula in disjunctive normal form. For example,

$$(x \vee y \vee \bar{z}) \wedge (\bar{x} \vee \bar{y}) \equiv (x \wedge \bar{y}) \vee (y \wedge \bar{x}) \vee (\bar{z} \wedge \bar{x}) \vee (\bar{z} \wedge \bar{y})$$

Now we can use the algorithm from part (a) to determine, in polynomial time, whether the resulting DNF formula is satisfiable. We have just solved 3SAT in polynomial time. Since 3SAT is NP-hard, we must conclude that $P = NP$!

10. (MEDIUM) Show that 3-SAT $\leq_p$ 4-SAT

11. (MEDIUM) Hamiltonian Path Problem (HP): Given a graph $G = (V, E)$, does there exist a path that visits each vertex exactly once?

Hamiltonian Cycle Problem (HC): Given a graph $G = (V, E)$, does there exist a cycle that visits each vertex exactly once and returns to the starting vertex?

Show that HP $\leq_p$ HC

12. (MEDIUM) INDEPENDENT SET: Is there an independent set of size $\geq k$ in G?
CLIQUE: Is there a clique of size $\geq k$ in G?

Show that INDEPENDENT SET $\leq$ CLIQUE and CLIQUE $\leq_p$ INDEPENDENT SET

13. (MEDIUM) In an undirected graph $G = (V, E)$, we say $D \subseteq V$ is a *dominating set* if every $v \in V$ is either in D or adjacent to at least one member of D. In the DOMINATING SET problem, the input is a graph and a budget b, and the aim is to find a dominating set in the graph of size at most b, if one exists.

Prove that this problem is NP-complete. [Hint: Assume that Vertex Cover is NP-Complete. Reduce Vertex Cover to Dominating Set]

14. (HARD) In task scheduling, it is common to use a graph representation with a node for each task and a directed edge from task i to task j if i is a precondition for j. This directed graph depicts the precedence constraints in the scheduling problem. Clearly, a schedule is possible if and only if the graph is acyclic. If it is not, we would like to identify the smallest number of constraints that must be dropped so as to make it acyclic.

Given a directed graph $G = (V, E)$, a subset $E' \subseteq E$ is called a **feedback arc set** if the removal of edges in E' renders G acyclic.

FEEDBACK ARC SET (FAS): Given a directed graph $G = (V, E)$ and a budget b, is there a feedback arc set of size $\leq b$. Show that FAS is NP-complete.